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A PARAMETRIC CLASS OF PRODUCTION STRATEGIES FOR MULTI-RESERVOIR PRODUCTION OPTIMIZATION

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Abstract

When a large oil or gas field is produced, several reservoirs often share the same processing facility. This facility is typically capable of processing only a limited amount of oil, gas and water per unit of time. In the present paper only single phase production, e.g., oil production, is considered. In order to satisfy the processing limitations, the production needs to be choked. That is, for each reservoir the production is scaled down by suitable *choke factors* between zero and one, chosen so that the total production does not exceed the processing capacity. Huseby & Haavardsson (2008) introduced the concept of a *production strategy*, a vector valued function defined for all points of time $t \geq 0$ representing the choke factors applied to the reservoirs at time t. As long as the total potential production rate is greater than the processing capacity, the choke factors should be chosen so that the processing capacity is fully utilized. When the production reaches a state where this is not possible, the production should be left unchoked. A production strategy satisfying these constraints is said to be *admissible*. Huseby & Haavardsson (2008) developed a general framework for optimizing production strategies with respect to various types of objective functions. In the present paper we present a parametric class of admissible production strategies. Using the framework of Huseby & Haavardsson (2008) it can be shown that under mild restrictions on the objective function an optimal strategy can be found within this class. The number of parameters needed to span the class is bounded by the number of reservoirs. Thus, an optimal strategy within this class can be found using a standard numerical optimization algorithm. This makes it possible to handle complex, high-dimensional cases. Furthermore, uncertainty may be included, enabling robustness and sensitivity analysis.

1 Introduction

Optimization is an important element in the management of multiple-field oil and gas assets, since many investment decisions are irreversible and finance is committed for the long term. Optimization of oil and gas recovery in petroleum engineering is a considerable research field, see Bittencourt & Horne (1997), Horne (2002) or Merabet & Bellah (2002). Another important research tradition focuses on the problem of modelling the entire hydrocarbon value chain, where the purpose is to make models for scheduling and planning of hydrocarbon field infrastructures with complex objectives, see van den Heever et al. (2001), Ivyer & Grossmann (1998) or Neiro & Pinto (2004). Since the entire value chain is very complex, many aspects of it need to be simplified to be able to construct such a comprehensive model.

Huseby & Haavardsson (2008) considered the more limited problem of hydrocarbon production optimization in an oil or gas field consisting of many reservoirs sharing the same processing facility. In order to satisfy the processing limitations of the facility, the production needs to be choked. That is, at any given point of time the production from each of the reservoirs are scaled down by suitable *choke factors* between zero and one, chosen so that the total production does not exceed the processing capacity. This situation was handled by introducing the concept of a *production strategy*. A production strategy is a vector valued function defined for all points of time $t \geq 0$ representing the choke factors applied to the reservoirs at time t. The problem is then reduced to finding a production strategy which is optimal with respect to a suitable objective function. Huseby & Haavardsson (2008) developed a general framework for solving such optimization problems, and provided solutions to the problem in several important special cases.

In the present paper we present a parametric class of production strategies. Using the framework of Huseby & Haavardsson (2008) it can be shown that under mild restrictions on the objective function an optimal strategy can be found within this class. The number of parameters needed to span the class is bounded by the number of reservoirs. Thus, an optimal strategy within this class can be found using a standard numerical optimization algorithm. This makes it possible to handle complex, high-dimensional cases. Furthermore, uncertainty may be included, enabling robustness and sensitivity analysis.

As in Huseby & Haavardsson (2008), we assume that the field has been analyzed using state-of-the-art reservoir simulation methods. Based on the output from these simulations simplified *production profile models* for each of the reservoirs can be constructed and used as input to the optimization procedure. How to construct such profile models is described in Haavardsson & Huseby (2007). We also follow the approach of Huseby & Haavardsson (2008) by focussing on optimizing oil production, and leave simultaneous analysis of oil, gas and water production for future work. Still the optimization methods developed here can be used in a broader context of a total value chain analysis.

2 Some basic concepts and established results

2.1 Model and notation

We consider the oil production from n reservoirs that share a processing facility with a constant process capacity $K > 0$, expressed in some suitable unit, e.g., kSm^3 per day. Let $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t))$ denote the vector of cumulative production functions for the n reservoirs, and let $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding vector of *potential production rate functions* or *PPR-functions*, where each PPR-function can be written as:

$$f_i(t) = f_i(Q_i(t)), \qquad t \geq 0, \ i = 1, \ldots n. \tag{2.1}$$

This assumption implies that the potential production rate can be expressed as a function of the volume produced. As a consequence the total producible volume from a reservoir does not depend on the production schedule. The assumption expressed in (2.1) also implies that the potential production rate of one reservoir does not depend on the volumes produced from the other reservoirs. We will also assume for $i = 1, \ldots n$ that f_i is non-negative and non-increasing as a function of $Q_i(t)$ for all t and that the recoverable volume of each reservoir is finite. Finally, to ensure uniqueness of potential production profiles we will also assume that f_i is Lipschitz continuous in Q_i, $i = 1, \ldots, n$.

A *production strategy* is defined by a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith reservoir at time t, $i = 1, \ldots, n$. The *actual production rates* from the reservoirs, after the production is choked is given by

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)),$$

where

$$q_i(t) = \frac{dQ_i(t)}{dt} = b_i(t) f_i(Q_i(t)), \quad i = 1, \ldots, n. \tag{2.2}$$

We also introduce the total production rate function $q(t) = \sum_{i=1}^{n} q_i(t)$ and the total cumulative production function $Q(t) = \sum_{i=1}^{n} Q_i(t)$. To reflect that $\boldsymbol{q}$ and $\boldsymbol{Q}$ depend on the chosen productions strategy $\boldsymbol{b}$, we sometimes indicate this by writing $\boldsymbol{q}(t) = \boldsymbol{q}(t, \boldsymbol{b})$ etc.

To satisfy the physical constraints of the reservoirs and the process facility, we require that the actual production rate cannot exceed its potential production rate. Moreover, the total production rate cannot exceed the production capacity. Let $\mathcal{B}$ denote the class of production strategies that satisfy these physical constraints. We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies*.

For a given production strategy $\boldsymbol{b} \in \mathcal{B}$ the *plateau length* is defined as

$$T_K = T_K(\boldsymbol{b}) = \sup\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t)) \geq K\}. \tag{2.3}$$

An *admissible production strategy* is defined as a production strategy $\boldsymbol{b} \in \mathcal{B}$ satisfying the following constraint:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{2.4}$$

Moreover, we let $\mathcal{B}' \subseteq \mathcal{B}$ denote the class of admissible strategies.

2.2 Objective functions

To evaluate production strategies we introduce an *objective function*, i.e., a mapping $\phi : \mathcal{B} \to \mathbb{R}$ representing some sort of a performance measure. If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$, we prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$ if $\phi(\boldsymbol{b}^2) \geq \phi(\boldsymbol{b}^1)$. Moreover, an *optimal production strategy* with respect to ϕ is a production strategy $\boldsymbol{b}^{opt} \in \mathcal{B}$ such that $\phi(\boldsymbol{b}^{opt}) \geq \phi(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$.

If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ are two production strategies such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$, one would most likely prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$. Thus, a sensible objective function should have the property that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$ whenever $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Objective functions satisfying this property will be referred to as *monotone objective functions*.

In general the revenue generated by the production may vary between the reservoirs. This may occur if e.g., the quality of the oil, or the average production cost per unit are different from reservoir to reservoir. Such differences should then be reflected in the chosen objective function. On the other hand, if all the reservoirs are similar, we could restrict ourselves to considering objective functions depending on the production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$). We refer to such objective functions as *symmetric*.

In this paper we will consider the following monotone, symmetric objective function:

$$\phi_{C,R}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du, \quad 0 \leq C \leq K, \quad R \geq 0. \tag{2.5}$$

The parameter R may be interpreted as a discount factor, while C is a threshold value reflecting the minimum acceptable production rate. If we insert $C = 0$ and $R > 0$ in (2.5), the resulting value of the objective function is simply the *discounted production.* On the other hand if we insert $C = K$ in (2.5), the integrand is positive only when $q(u) = K$. When $R = 0$ we obtain that $\phi_{C,0}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. It also follows from the definition of $\phi_{C,R}$ in (2.5) and T_K in (2.3) that $\phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b}) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b}))$.

2.3 Principles for optimizing production strategies

We now turn to the problem of finding the best production strategy. Consider the set

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n], \tag{2.6}$$

where $V_1, \ldots, V_n$ are the recoverable volumes from the n reservoirs. We then introduce the subset $\mathcal{M} \subseteq \mathcal{Q}$ given by:

$$\mathcal{M} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \geq K\}.$$

The subset $\bar{\mathcal{M}} \subseteq \mathcal{Q}$ is defined as $\bar{\mathcal{M}} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) < K\}$. We also need the set of boundary points of $\mathcal{M}$ separating $\mathcal{M}$ from $\bar{\mathcal{M}}$, which we denote by $\partial(\mathcal{M})$. Thus, $\boldsymbol{Q} \in \partial(\mathcal{M})$ if and only if every neighborhood of $\boldsymbol{Q}$ intersects both $\mathcal{M}$ and $\bar{\mathcal{M}}$.

The set $\mathcal{M}$ has the property that the total production rate can be sustained at plateau level as long as $\boldsymbol{Q} \in \mathcal{M}$. More specifically, let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$ as t increases. From the boundary conditions we know that $\boldsymbol{Q}(0) = \boldsymbol{0}$. By the continuity of the PPR-functions, $\boldsymbol{Q}(t)$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.

If $\boldsymbol{b} \in \mathcal{B}$, the resulting path is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}'$, the path is called an *admissible path.* in general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}'$. Moreover, we let $\partial(\mathcal{M}') = \partial(\mathcal{M}) \cap \mathcal{M}'$.

For an admissible path the total production rate equals K all the way until the path reaches $\partial(\mathcal{M}')$. Moreover, the plateau length $T_K(\boldsymbol{b})$ is the point in time when the path reaches $\partial(\mathcal{M}')$, implying that $\partial(\mathcal{M}') = \{\boldsymbol{Q}(T_K(\boldsymbol{b})) : \boldsymbol{b} \in \mathcal{B}'\}$.

The following proposition, proved in Huseby & Haavardsson (2008), plays a key role when searching for optimal production strategies:

Proposition 2.1 *Let ϕ be a symmetric, monotone objective function and let $\boldsymbol{b} \in \mathcal{B}'$. Then ϕ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$. Thus, we may write $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$.*

As a consequence of Proposition 2.1 the following corollary can be stated:

Corollary 2.2 *Let ϕ be a symmetric, monotone objective function and let $\boldsymbol{b} \in \mathcal{B}'$ and let $\boldsymbol{Q}^* \in \partial\mathcal{M}'$ denote the point with the property that $\phi(\boldsymbol{Q})$ is maximized for $\boldsymbol{Q} = \boldsymbol{Q}^*$. Assume that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^*$. Then $\boldsymbol{b}$ is optimal with respect to ϕ.*

Corollary 2.2 states that any admissible production strategy which path reaches the optimal $\boldsymbol{Q}^*$ is optimal. The following corollary will be useful in the present paper:

Corollary 2.3 *Let ϕ be a symmetric, monotone objective function and let $\mathcal{C} \subseteq \mathcal{B}'$ be a class of admissible production strategies such that for all $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ there exists a $\boldsymbol{b}^* \in \mathcal{C}$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b}^*)) = \boldsymbol{Q}^*$. Then an optimal production strategy with respect to ϕ can always be found within $\mathcal{C}$.*

Motivated by Corollary 2.2 Huseby & Haavardsson (2008) proposed a two- step process for finding an optimal production strategy. The first step consisted of finding $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ such that $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M}')$. In the second step a *backtracking algorithm* was used to derive a production strategy $\boldsymbol{b}^* \in \mathcal{B}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^*$ which by Corollary 2.2 is optimal.

If all the PPR-functions are differentiable, the first step can often be solved very efficiently using e.g., the method of Lagrange multipliers. Using such a method one can at least find a $\boldsymbol{Q}' \in \partial(\mathcal{M})$ such that $\phi(\boldsymbol{Q}') \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M})$. If $\boldsymbol{Q}' \in \partial(\mathcal{M}')$ as well, we let $\boldsymbol{Q}^* = \boldsymbol{Q}'$. To verify that $\boldsymbol{Q}' \in \partial(\mathcal{M}')$, Huseby & Haavardsson (2008) uses the backtracking algorithm. If this algorithm successfully produces an admissible path, this shows that $\boldsymbol{Q}' \in \partial(\mathcal{M}')$. Thus, an optimal production strategy is found. On the other hand, if $\boldsymbol{Q}' \in \partial(\mathcal{M} \setminus \mathcal{M}')$, no such admissible path exists. Thus, the backtracking algorithm cannot succeed. Note, however, that even if $\boldsymbol{Q}' \in \partial(\mathcal{M}')$, the backtracking algorithm may sometimes fail. This occurs when $\boldsymbol{Q}'$ is very close to or at the border of $\partial(\mathcal{M}')$.

In the next section we propose an alternative approach to the optimization problem. We introduce a parametric class of admissible production strategies and restrict our search for optimal strategies within this class. If all $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ can be reached using production strategies from this parametric class, Corollary 2.3 then states that an optimal production strategy can always be found within this class.

3 A parametric class of production strategies

A simple production strategy can always be constructed by using the same choke factor for all the reservoirs. That is, we let $b_i(t) = c(t)$, $i = 1, \ldots, n$. For such a production strategy to be admissible $c(t)$ must satisfy the following:

$$\sum_{i=1}^{n} c(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{3.1}$$

Thus, for $0 \leq t \leq T_K$, we have:

$$c(t) = \frac{K}{\sum_{i=1}^{n} f_i(Q_i(t))}, \tag{3.2}$$

while $c(t) = 1$ for all $t > T_K$. Note that since $\sum_{i=1}^{n} f_i(Q_i(t)) \geq K$ for $0 \leq t \leq T_K$, the common choke factor, $c(t)$ will always be less than or equal to 1. A production strategy defined in this way, will be referred to as a *symmetry strategy*. We observe that when a symmetry strategy is used, the available production capacity is shared among the reservoirs such that none of the reservoirs are given any kind of priority. The idea now is to expand this class by allowing the production capacity to be shared asymmetrically. To facilitate this we start out by considering production strategies where for $0 \leq t \leq T_K$ the choke factors are given by:

$$b_i(t) = w_i c(t), \qquad i = 1, \ldots, n, \tag{3.3}$$

where $w_1, \ldots, w_n$ are positive real numbers representing the relative priorities assigned to the n reservoirs, and where $c(t)$ is chosen so that the strategy is admissible. For $t > T_K$, we of course define $b_i(t) = 1$, $i = 1, \ldots, n$. Note that if $w_1 = \cdots = w_n$ we get a symmetry strategy.

In order to ensure admissibility, $c(t)$ must be chosen so that:

$$\sum_{i=1}^{n} w_i c(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{3.4}$$

Thus, for $0 \leq t \leq T_K$ the choke factors are given by:

$$b_i(t) = w_i c(t) = \frac{w_i K}{\sum_{j=1}^{n} w_j f_j(Q_j(t))}, \qquad i = 1, \ldots, n. \tag{3.5}$$

Unfortunately, this definition does not guarantee that the choke factors are less than or equal to 1. To fix this problem, we instead let:

$$b_i(t) = \min\{1, w_i c(t)\}, \qquad i = 1, \ldots, n. \tag{3.6}$$

While this ensures that the resulting production strategy is valid, it makes the calculation of $c(t)$ slightly more complicated. To ensure admissibility, $c(t)$ must now be chosen so that:

$$\sum_{i=1}^{n} \min\{1, w_i c(t)\} f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{3.7}$$

When $t > T_K$, $c(t)$ must be chosen large enough so that $\min\{1, w_i c(t)\} = 1$, $i = 1, \ldots, n$. One obvious possibility is to let $c(t) = \max_i\{w_i^{-1}\}$. When $0 \leq t \leq T_K$, there is always a unique value of $c(t)$ satisfying (3.7). To see this we first note that if we let $c(t) = 0$, the left-hand side of (3.7) is zero which is less than K. On the other hand, letting $c(t) = \max\{w_i^{-1}\}$, we get that $\min\{1, w_i c(t)\} = 1$, $i = 1, \ldots, n$. Inserting this, the left-hand side of (3.7) becomes $\sum_{i=1}^{n} f_i(Q_i(t))$, which is greater than or equal to K for $0 \leq t \leq T_K$. Between these two extremes the left-hand side of (3.7) is a continuous function of $c(t)$. Thus, the existence of a $c(t)$ satisfying (3.7) is guaranteed by the intermediate value theorem. Moreover, since the left-hand side of (3.7) is a strictly increasing function of $c(t)$, this $c(t)$-value is unique.

In order to take a closer look at the calculation of $c(t)$ for the case where $0 \leq t \leq T_K$, it is convenient to sort the weights in decreasing order. This can always be done by identifying a permutation π of the index set so that $w_{\pi(1)} \geq w_{\pi(2)} \geq \cdots \geq w_{\pi(n)}$. In order to simplify the notation, however, we instead assume that the reservoirs are indexed so that $w_1 \geq w_2 \geq \cdots \geq w_n$. We then introduce the following sets:

$$\mathcal{M}_k = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{k} f_i(Q_i) + \sum_{i=k+1}^{n} \frac{w_i}{w_k} f_i(Q_i) \geq K\}, \qquad k = 1, \ldots, n. \tag{3.8}$$

We also define $\mathcal{M}_0 = \emptyset$. We observe that the left-hand side of the inequality defining the $\mathcal{M}_k$, is a weighted sum of the PPR-functions, where the weight associated to f_i is 1 for $i = 1, \ldots, k$ and $\frac{w_i}{w_k}$ for $i = k+1, \ldots, n$. Moreover, since $w_1 \geq w_2 \geq \cdots \geq w_n$, it follows that $\frac{w_i}{w_k} \leq 1$ for $i = k+1, \ldots, n$. As k increases, the number of PPR-functions with weight 1 increases. At the same time the weights of form $\frac{w_i}{w_k}$ increases as well. Thus, the left-hand side of the inequality defining the set $\mathcal{M}_k$ increases with k. Hence, it follows that $\mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}_n$. When $k = n$, all the PPR-functions have weight 1, implying that $\mathcal{M}_n = \mathcal{M}$. From this it follows that the difference sets $(\mathcal{M}_k \setminus \mathcal{M}_{k-1})$, $k = 1, \ldots, n$, form a partition of the set $\mathcal{M}$. Note, however, that if $w_k = w_{k-1}$, then $\mathcal{M}_k = \mathcal{M}_{k-1}$. Thus, some of the difference sets may be empty.

In order to find the $c(t)$-function satisfying (3.7) when $0 \le t \le T_K$, i.e., when $\boldsymbol{Q}(t) \in \mathcal{M}$, it is convenient to solve this problem separately for each of the difference sets. Thus, we let $\boldsymbol{Q}(t) \in \mathcal{M}_k \setminus \mathcal{M}_{k-1}$, and claim that in this case $c(t)$ is given by:

$$c(t) = \frac{K - \sum_{i<k} f_i(Q_i(t))}{\sum_{i=k}^n w_i f_i(Q_i(t))}. \tag{3.9}$$

To prove this we note that since $\boldsymbol{Q}(t) \in \mathcal{M}_k$, it follows that:

$$\sum_{i=1}^k f_i(Q_i(t)) + \sum_{i=k+1}^n \frac{w_i}{w_k} f_i(Q_i(t)) \ge K. \tag{3.10}$$

By multiplying both sides by w_k and rearranging the terms we obtain that:

$$\frac{w_k(K - \sum_{i<k} f_i(Q_i(t)))}{\sum_{i=k}^n w_i f_i(Q_i(t))} \le 1. \tag{3.11}$$

Combining this with (3.9) we get that $w_k c(t) \le 1$, and since $w_k \ge w_{k+1} \ge \cdots \ge w_n$, it follows that:

$$w_i c(t) \le 1, \qquad i = k, \dots, n. \tag{3.12}$$

On the other hand we have that $\boldsymbol{Q}(t) \notin \mathcal{M}_{k-1}$. Thus, by using a similar argument as above, we get that:

$$w_i c(t) \ge 1, \qquad i = 1, \dots, k-1. \tag{3.13}$$

Hence, it follows that:

$$b_i(t) = \min\{1, w_i c(t)\} = \begin{cases} 1 & i = 1, \dots, k-1 \\ w_i c(t) & i = k, \dots, n \end{cases} \tag{3.14}$$

By inserting (3.9) and (3.14) into the left-hand side of (3.7) we get that:

$$\begin{aligned} \sum_{i=1}^n \min\{1, w_i c(t)\} f_i(Q_i(t)) &= \sum_{i<k} f_i(Q_i(t)) + \sum_{i=k}^n w_i c(t) f_i(Q_i(t)) \\ &= \sum_{i<k} f_i(Q_i(t)) + \frac{[K - \sum_{i<k} f_i(Q_i(t))] \sum_{i=k}^n w_i f_i(Q_i(t))}{\sum_{i=k}^n w_i f_i(Q_i(t))} \\ &= K. \end{aligned}$$

That is, $c(t)$ as given in (3.9), satisfies (3.7) when $\boldsymbol{Q}(t) \in \mathcal{M}_k \setminus \mathcal{M}_{k-1}$. Since the same argument holds for all $k = 1, \dots, n$, it follows that $c(t)$ satisfies (3.7) for all $\boldsymbol{Q}(t) \in \mathcal{M}$, i.e., whenever $0 \le t \le T_K$ as claimed.

By varying the weights $w_1, \dots, w_n$ in $\mathbb{R}_+^n$ a whole range of admissible production strategies is obtained. We will refer to such production strategies as *first-order fixed-weight strategies*, and denote the class of all such strategies by $\mathcal{B}_1^w$. We always assume that the corresponding $c(t)$ is determined by (3.9) ensuring that the resulting production strategy is admissible. Thus, $\mathcal{B}_1^w \subseteq \mathcal{B}'$. If $\boldsymbol{b} \in \mathcal{B}_1^w$ is a fixed-weight strategy with weight vector $\boldsymbol{w} = (w_1, \dots, w_n)$, we sometimes indicate this by writing $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{w})$.

From the formula (3.9) it is easy to see that if we replace the weight vector $\boldsymbol{w}$ by $\tilde{\boldsymbol{w}} = \lambda \boldsymbol{w}$ where $\lambda > 0$, then $c(t)$ is replaced by $\tilde{c}(t) = \lambda^{-1} c(t)$. As a result the choke factors are not affected by this change of weights. Thus, we have shown that:

$$\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}(\lambda \boldsymbol{w}). \tag{3.15}$$

That is, the production strategy is invariant with respect to scale transformations of the weight vector $\boldsymbol{w}$. This means that one can reduce the dimension of the space of possible weight vectors to $(n-1)$ without changing the class $\mathcal{B}_1^w$. There are several ways of doing this. One possibility is to consider only $\boldsymbol{w}$ of length 1. Another possibility is to restrict the search to $\boldsymbol{w}$ normalized so that the sum of weights is 1. Here, however, we have chosen a third option, where the dimension is reduced by fixing the value of one of the weights, e.g., by letting $w_n = 1$. All the remaining weights may be chosen as arbitrary positive real numbers. Thus, the resulting search area is the unbounded $(n-1)$-dimensional set $\mathbb{R}_+^{n-1}$. Sometimes, however, it is easier to carry out the search on a bounded set. This can be achieved by using the following reparametrization:

$$w_i = \frac{v_i}{1-v_i}, \qquad i = 1, \ldots, (n-1). \tag{3.16}$$

By letting v_i run through all values in the interval $(0,1)$, the resulting values of w_i will run through all positive real numbers. Thus, searching for the optimal values of $(w_1, \ldots, w_{n-1})$ within the unbounded set $\mathbb{R}_+^{n-1}$ is reduced to searching for the optimal values of $(v_1, \ldots, v_{n-1})$ within the bounded set $(0,1)^{n-1}$. This reparametrization is used in the numerical examples discussed in Section 4.

3.1 Higher order fixed-weight strategies

A weakness with the class $\mathcal{B}_1^w$ is that it does not allow strict priorities between the reservoirs. In order to study this further we introduce the concept of a *priority strategy*. A *kth order priority strategy* is an admissible production strategy defined relative to an ordered partition $\{A_j\}_{j=1}^k$ of the index set $\{1, \ldots, n\}$ of the reservoirs. The available processing capacity K is divided between the n reservoirs so that the reservoirs in A_1 are given the highest priority, the reservoirs in A_2 are given the second highest priorities, and so on. More specifically, at any given point of time t we let $K_j(t)$ denote the processing capacity available to the reservoirs in A_j, $j = 1, \ldots, k$. Then:

$$\begin{aligned}
K_1(t) &= K, \qquad (3.17)\\
K_2(t) &= \max\{0, K_1(t) - \sum_{i \in A_1} f_i(Q_i(t))\},\\
K_3(t) &= \max\{0, K_2(t) - \sum_{i \in A_2} f_i(Q_i(t))\},\\
&\cdots\\
K_k(t) &= \max\{0, K_{k-1}(t) - \sum_{i \in A_{k-1}} f_i(Q_i(t))\}.
\end{aligned}$$

In order to ensure admissibility it is assumed that the reservoirs in all groups use as much as possible of the available processing capacity. Thus, the choke factors $b_1(t), \ldots, b_n(t)$ are chosen so that:

$$\sum_{i \in A_j} b_i(t) f_i(Q_i(t)) = \min\{K_j(t), \sum_{i \in A_j} f_i(Q_i(t))\}, \qquad j = 1, \ldots, k. \tag{3.18}$$

Note that that (3.18) implies that the production strategy is admissible since adding up all k equalities yields:

$$\sum_{i=1}^n b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^n f_i(Q_i(t))\}. \tag{3.19}$$

The most extreme type of a priority strategy is an nth order priority strategy. For such strategies $|A_j| = 1$ for $j = 1, \dots, n$. Thus, the ordered partition, $\{A_j\}_{j=1}^n$, simply represents a permutation of the reservoirs. In this case the production strategy is uniquely defined by this permutation. Thus, if $A_j = \{i_j\}$, $j = 1, \dots, n$, then:

$$K_j(t) = \max\{0, K_{j-1}(t) - f_{i_{j-1}}(Q_{i_{j-1}}(t))\}, \qquad j = 2, \dots n, \tag{3.20}$$

while the choking factors, $b_1(t), \dots, b_n(t)$, satisfies:

$$b_{i_j}(t) = \min\{1, \frac{K_j(t)}{f_{i_j}(Q_{i_j}(t))}\}, \qquad j = 1, \dots n. \tag{3.21}$$

We now proceed by combining fixed-weight strategies and priority strategies. Thus, we let $\{A_j\}_{j=1}^k$ be a partition of the index set, and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be a vector of positive real numbers. We then consider choke factor functions of the form:

$$b_i(t) = \min\{1, w_i c_j(t)\}, \qquad i \in A_j,\ j = 1, \dots, k, \tag{3.22}$$

where $c_1(t), \dots, c_k(t)$ are determined for each t so that the resulting production strategy is admissible., i.e., so that:

$$\sum_{i \in A_j} \min\{1, w_i c_j(t)\} f_i(Q_i(t)) = \min\{K_j(t), \sum_{i \in A_j} f_i(Q_i(t))\}, \qquad j = 1, \dots, k. \tag{3.23}$$

A production strategy of this form will be referred to as a *kth order fixed-weight strategy*, and we denote the class of all such strategies by $\mathcal{B}_k^w$. Computing $c_1(t), \dots, c_k(t)$ can be done in exactly the same way as for first-order fixed-weight strategies, so we skip the details here.

If $\boldsymbol{b} \in \mathcal{B}_k^w$ is a kth order fixed-weight strategy relative to the partition $\{A_j\}_{j=1}^k$ and with weight vector $\boldsymbol{w} = (w_1, \dots, w_n)$, we introduce the following vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$, where $\boldsymbol{w}_j$ is obtained from $\boldsymbol{w}$ by replacing w_i by 0 for all $i \notin A_j$, $j = 1, \dots, k$. Thus, since $A_1, \dots, A_k$ are pairwise disjoint and $A_1 \cup \dots \cup A_k = \{1, \dots, n\}$, it follows that $\boldsymbol{w} = \boldsymbol{w}_1 + \dots + \boldsymbol{w}_k$. Now, let $\lambda_1, \dots, \lambda_k$ be positive numbers, and assume that the vector of weights, $\boldsymbol{w}$, is replaced by $\tilde{\boldsymbol{w}} = \sum_{j=1}^k \lambda_j \boldsymbol{w}_j$. Then, using the same argument as in the first-order case, it follows that $c_1(t), \dots, c_k(t)$ are replaced by $\tilde{c}_1(t), \dots, \tilde{c}_k(t)$, where $\tilde{c}_j(t) = \lambda_j^{-1} c(t)$. As a result the choke factors are not affected by this change of weights. Thus, we have shown that:

$$\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}(\sum_{j=1}^n \lambda_j \boldsymbol{w}_j). \tag{3.24}$$

This implies that we in the kth order case may reduce the dimension of the space of possible weight vectors to $(n-k)$ without reducing the class $\mathcal{B}_k^w$. We have chosen to do this by fixing the value of one weight for each of the sets $A_1, \dots, A_k$. In the case where $k = n$, we know that the priority strategy is uniquely determined by the permutation given by the single element sets $A_1, \dots, A_n$. Thus, in this case the weight vector does not affect the production strategy, which is reflected by the fact that the dimension of the space of possible weight vectors can be reduced to zero.

We recall that by Corollary 2.3 an optimal production strategy can be found within a given class of admissible strategies provided that all points in $\partial(\mathcal{M}')$ can be reached by members of this class. It turns out that all *interior* points of $\partial(\mathcal{M}')$ can be reached by first-order fixed-weight strategies. However, to reach the boundary points in $\partial(\mathcal{M}')$ as well, higher-order strategies must be included. In a forthcoming paper it will be proved that by considering the combined class of fixed-weight strategies of *all* orders, it is possible to reach all points in $\partial(\mathcal{M}')$. Hence, an optimal production strategy can always be found within $\mathcal{B}_1^w \cup \cdots \cup \mathcal{B}_n^w$.

Assuming that the value of the objective function, ϕ, interpreted as a function of $\boldsymbol{Q}(T_K(\boldsymbol{b}))$, is a continuous function of this vector, it follows that for each point $\boldsymbol{Q}^* \in \partial(\partial(\mathcal{M}'))$ and $\epsilon > 0$, there exists another point, $\tilde{\boldsymbol{Q}}$ in the interior of $\partial(\mathcal{M}')$ such that $|\phi(\boldsymbol{Q}^*) - \phi(\tilde{\boldsymbol{Q}})| < \epsilon$. Hence, even if the search for an optimal strategy is restricted to $\mathcal{B}_1^w$, it is possible to find a strategy which is approximately optimal. In order to approximate a higher order fixed-weight strategy by a first-order strategy, one can assign very high weights to the reservoirs in the set with highest priority, and then use significantly smaller weights for the reservoirs in the sets with lower priorities. As we shall see, however, if the optimal strategy is a higher order strategy, better numerical results are obtained by searching among the fixed-weight strategies with the correct order.

4 Numerical optimization

We will now describe how the objective function $\phi_{C,R}(\boldsymbol{b})$ defined in (2.5) can be maximized for the parametric class defined in Section 3 using numerical optimization techniques. The numerical Java library Java Tools for Experimental Mathematics (JTEM)[1] is used for the optimization.

4.1 Initialization

Case studies have suggested that the objective function $\phi_{C,R}(\boldsymbol{b}(\boldsymbol{w}))$ is multi-modal as a function of $\boldsymbol{w}$. To ensure that the global maximum is found, several initialization techniques may be used, see Liu (2001). We have chosen to initialize the search by sampling N n-dimensional vectors at random. Then the objective function is evaluated in all N vectors. The vector where the object function attains its maximum, constitutes the initialization vector for the numerical search.

4.2 A sequential approach for numerical optimization

To find the optimal strategy within $\mathcal{B}_1^w \cup \cdots \mathcal{B}_1^w$ we need to identify the correct order of the optimal strategy. Moreover, for a given order, say k, we need an algorithm for finding the optimal strategy within $\mathcal{B}_k^w$. We start out by presenting the last algorithm:

Algorithm 4.1 *Let ϕ be a monotone, symmetric objective function. Assume that an ordered partition $\{A_j\}_{j=1}^k$ is given. Denote the highest element in each A_j with i_{A_j}, $j = 1, \ldots, k$. Then a production strategy $\boldsymbol{b}^* \in \mathcal{B}_k^w$ which maximizes ϕ numerically can be found as follows:*

STEP 1. *Find N random samples of $\boldsymbol{w}_k$ using the techniques described in Section 4.1. We set $w_{i_{A_j}} = 1.0$ for $j = 1, \ldots, k$ to avoid over-parametrization, as explained in Section 3.1. Denote these samples $\boldsymbol{w}_k^1, \ldots, \boldsymbol{w}_k^N$. Among these we select a vector $\boldsymbol{w}_k^j$ such that*

$$\phi(\boldsymbol{b}(\boldsymbol{w}_k^j)) \geq \phi(\boldsymbol{b}(\boldsymbol{w}_k^i)), \text{ for all } i \in \{1, \ldots, N\}.$$

[1]For documentation see `http://www.jtem.de/`.

Step 2. *Maximize ϕ numerically with respect to $\boldsymbol{w}$ using $\boldsymbol{w}_k^j$ as initialization vector. In the maximization we always keep $w_{i_{A_j}} = 1.0$ for $j = 1, \ldots, k$. Denote the resulting vector of weights $\boldsymbol{w}_k^*$.*

To find the correct order of the optimal strategy we start by searching among the first-order fixed-weight strategies, and denote by $\boldsymbol{w}_1^*$ the resulting candidate obtained from Algorithm 4.1. Assuming that ϕ is continuous, it follows, as explained in Section 3.1, that $\boldsymbol{w}_1^*$ will be approximately optimal. We then proceed by inspecting $\boldsymbol{w}_1^*$. If the ratio between the smallest and largest element of this vector is large, this indicates that the optimal strategy may be a higher order strategy. Thus, the natural next step is to consider second order fixed-weight strategies. Prior to this we sort the elements of $\boldsymbol{w}_1^*$ so that

$$w_{1,i_1}^* \geq \cdots \geq w_{1,i_n}^*.$$

Since this ordering indicates a prioritization order of the reservoirs, we consider only second order fixed-weight strategies such that the weights corresponding to the indices in A_1 are larger than the weights corresponding to the indices in A_2. Thus, only the following $n-1$ partitions need to be considered:

$$A_1 = \{i_1\}, \quad A_2 = \{i_2, \ldots, i_n\}, \tag{4.1}$$

$$A_1 = \{i_1, i_2\}, \quad A_2 = \{i_3, \ldots, i_n\}, \tag{4.2}$$

$$\cdots$$

$$A_1 = \{i_1, \ldots, i_{n-1}\}, \quad A_2 = \{i_n\}. \tag{4.3}$$

$$\tag{4.4}$$

We then run Algorithm 4.1 for all these partitions and denote by $\boldsymbol{w}_2^*$ the best-performing weight vector. If $\phi(\boldsymbol{b}(\boldsymbol{w}_2)) < \phi(\boldsymbol{b}(\boldsymbol{w}_1^*))$ we use $\boldsymbol{w}_1^*$ and conclude that the optimal strategy is a first-order strategy. Otherwise we proceed using $\boldsymbol{w}_2^*$ and the corresponding partition instead of $\boldsymbol{w}_1^*$. We then inspect the two sub-vectors of $\boldsymbol{w}_2^*$ corresponding to A_1 and A_2. If the ratio between the smallest and largest element of any of these two sub-vectors is large, this indicates that the optimal strategy may be an even higher order strategy. We then proceed by considering third order strategies. Now, however, only refinements of the previous partitions are considered. This implies that only $n-2$ partitions need to be examined at this stage. The process is repeated until no further improvement can be obtained.

By only considering successive refinements of the previous partitions, and taking into account the ordering of the weights, the number of times we need to run Algorithm 4.1 is reduced to a minimum. Thus, the total order of the sequential optimization process is dominated by the order of this algorithm.

5 Examples

5.1 The fixed-weight strategy as an alternative to backtracking

In this first example we will assume that both steps of the optimization algorithm developed in Huseby & Haavardsson (2008) can be executed. We start with finding the optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs. Then we use backtracking to derive an admissible production strategy to reach $\boldsymbol{Q}^*$. Corollary 2.2 states that any admissible production strategy which path reaches the optimal $\boldsymbol{Q}^*$ is optimal. Thus, as an alternative to backtracking we use the proposed parametric class to find another admissible production strategy to reach $\boldsymbol{Q}^*$.

To find $\boldsymbol{Q}^*$ we use the theory of Huseby & Haavardsson (2008), which states that if all PPR-functions are concave, the optimal $\boldsymbol{Q}^*$ may typically be located in central parts of $\partial(\mathcal{M}')$. The PPR-functions $f_1, \ldots, f_n$ are given by

$$f_i(Q_i(t)) = \sqrt{D_i(V_i - Q_i(t))},\ i = 1, \ldots, n, \tag{5.1}$$

where $V_1, \ldots, V_n$ denote the recoverable volumes of the n reservoirs. The chosen parameter values of the example are listed in Table 5. The objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (2.5) is used. As explained in Section 2.2 the optimal solution maximizes the plateau volume, $\phi_{K,0}(\boldsymbol{Q}) = \sum_{i=1}^n Q_i$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$. When all PPR-functions are concave, an optimal solution to the first step of the optimization algorithm developed in Huseby & Haavardsson (2008) typically involves finding the separating hyperplane supporting $\mathcal{M}$ at the optimal $\boldsymbol{Q}^*$. Further, we realize that the PPR-functions on the form given by (5.1) and the extended objective function $\phi_{K,0} = \sum_{i=1}^n Q_i$ are differentiable, so Lagrange multipliers may be used. Using Lagrange multipliers it is straight-forward to show that the optimal $\boldsymbol{Q}^*$, denoted $\boldsymbol{Q}^*_L$, is given by

$$\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*) = (V_1 - \frac{D_1}{2}\{\frac{K}{\sum_{i=1}^n D_i}\}^2, \ldots, V_n - \frac{D_n}{2}\{\frac{K}{\sum_{i=1}^n D_i}\}^2). \tag{5.2}$$

To compare $\boldsymbol{Q}^*_L$ with the boundary point $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ obtained using the best first-order fixed-weight strategy we calculate $\boldsymbol{b}^* \in \mathcal{B}_1^w$, as explained in Section 4.2. Table 1 lists the coordinates of $\boldsymbol{Q}^*_L$ and $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ for the example of this section and the example of Section 5.2. Table 2 correspondingly lists the objective function values for $\boldsymbol{Q}^*_L$ and $\boldsymbol{Q}^*_P$ and the Euclidian distance between these two points. From Table 2 we observe that the distance between $\boldsymbol{Q}^*_L$ and $\boldsymbol{Q}^*_P$ is small, as expected. Table 3 lists the weights of the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$. As we can see from Table 3 none of the weights are significantly larger than the others, indicating that the optimum $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ is an interior point of $\partial(\mathcal{M})$. When we execute Algorithm 4.1 with second-order fixed-weight strategies this is confirmed; we are not able to find any second-order fixed-weight strategy with the property that ϕ_K is increased. Thus we conclude $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ is an interior point of $\partial(\mathcal{M}')$. The fact that the backtracking algorithm manages to propose an admissible production strategy to reach $\boldsymbol{Q}^*_L$ also indicates that $\boldsymbol{Q}^*_L$ is an interior point, see Huseby & Haavardsson (2008) for a discussion.

Example	Boundary point	Reservoir					
		1	2	3	4	5	6
Section 5.1	$\boldsymbol{Q}^*_L$	3,619.6	4,459.1	5,454.4	n.a.	n.a.	n.a.
	$\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$	3,619.2	4,458.4	5,453.9	n.a.	n.a.	n.a.
Section 5.2	$\boldsymbol{Q}^*_L$	3,879.2	4,731.7	5,896.0	5,275.6	7,832.3	8,141.2
	$\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$	3,881.7	4,747.0	5,968.2	5,273.6	7,673.3	8,190.4

Table 1: *Coordinates of the points* $\boldsymbol{Q}^*_L, \boldsymbol{Q}^*_{\boldsymbol{w}_1^*} \in \partial(\mathcal{M}')$ *in the examples in the sections 5.1 and 5.2.*

Proceeding to the backtracking, Figure 1 shows the production rates of this example when backtracking is used. The backtracking algorithm follows a piecewise linear path from the

Example	Process constraint (in kSm^3 per sd)	$\phi_K(\boldsymbol{Q}_L^*)$ (in kSm^3)	$\phi_K(\boldsymbol{Q}_P^*)$ (in kSm^3)	Distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_P^*$ (in kSm^3)
Section 5.1	3.0	13,531.5	13,533.1	1.3
Section 5.2	7.0	35,756.2	35,737.3	182.1

Table 2: *Comparison of $\phi_K(\boldsymbol{Q}_L^*)$ and $\phi_K(\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*)$ and the distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_P^*$ in the examples of the sections 5.1 and 5.2.*

Example Section	Best first-order fixed-weight strategy, $\boldsymbol{w}_1^*$	$\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$ (kSm^3)	Best higher-order fixed-weight strategy, $\{A_j\}_{j=1}^k$ and $\boldsymbol{w}_k^*$	$\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_k^*))$ (kSm^3)	k
5.1	$(2.28, 2.0, 1.0)$	13,533.1	n.a.	n.a	n.a.
5.2	$(2.74, 1.56, 0.67, 0.82, 44.73, 1.0)$	35,737.3	$\{5\}$, $\{1,2,3,4,6\}$ $(2.75, 1.56, 0.67, 0.82, 1.0, 1.0)$	35,737.3	2
5.3	$(0.047, 1.141, 0.036, 0.028, 16.34, 12.28, 1.41, 1.0)$	31,856.2	$\{5\},\{6\},\{7\},\{8\}$, $\{1,2,3,4\}$ $(1.54, 1.48, 1.19, 1.0, 1.0, 1.0, 1.0, 1.0)$	32,230.3	5

Table 3: *The best first-order fixed-weight strategies, and the best higher-order fixed-weight strategies if the optimum is in the boundary of $\partial(\mathcal{M}')$ in the examples of the sections 5.1, 5.2 and 5.3.*

optimal $\boldsymbol{Q}^*$ to $\mathbf{0}$. At distinct time points the actual production is found using the well-known Simplex algorithm, see Huseby & Haavardsson (2008) for details. Due to the extreme nature of this algorithm, the production rates of the individual reservoirs tend to oscillate in periods. The oscillation occurs when it is equally beneficial to produce from two or more reservoirs, so that when the reservoirs compete for capacities they will alternate between being produced in one period and choked the next. If the primary purpose of the production optimization is to give decision support to project teams, the oscillations are not critical. The focus can for example be the assessment of different infrastructure investment alternatives. Hence, we are interested in the resulting cash flows of these different alternatives so that we can ultimately select and recommend one of the alternatives. The purpose is not to give the obtained production strategy as an input for long-term production planning to a field manager. In a real production setting it would not be advisable to produce the reservoirs as prescribed by the backtracking algorithm, due to the oscillations of the individual production rates.

Figure 2 shows the production rates of this example when the proposed parametric class is used. The production rates from the proposed parametric class yield smooth, interpretable production rates that do not fluctuate. These production rates can also be used in decision support, as explained above. In addition the production rates can be used in long-term

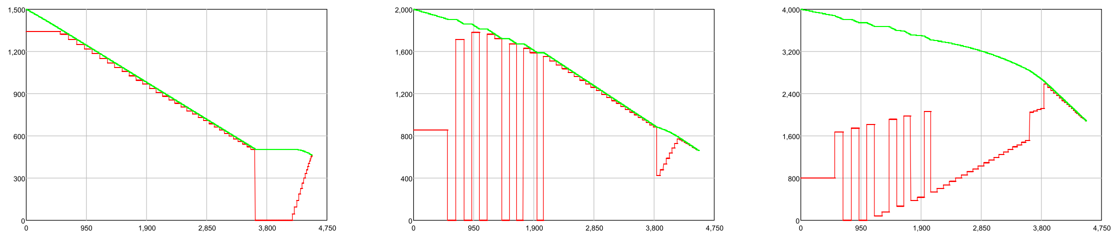

Figure 1: *The red graphs display the actual production rates when the backtracking algorithm proposed in Huseby & Haavardsson (2008) is used. The green graphs display the potential production rate functions.*

planning of the actual production of a field consisting of many reservoirs. In particular the production strategy can be used to assist production managers when they want to coordinate the production of many reservoirs. Furthermore, the proposed parametric class is better suited than the backtracking algorithm for feedback to the reservoir simulation team on possible modifications of the simulations. Hence, the proposed parametric class serves multiple purposes.

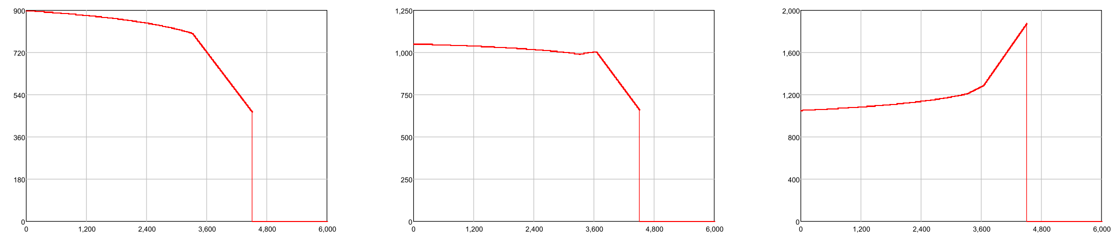

Figure 2: *The actual production rates when the proposed parametric class is used.*

5.2 A case where backtracking fails

In the second example we consider an example where we are able to execute the first step but not the second step of the two step optimization algorithm developed in Huseby & Haavardsson (2008). Thus, we find the optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs, but the backtracking algorithm fails to propose an admissible production strategy to reach $\boldsymbol{Q}^*$. As explained in Huseby & Haavardsson (2008), this indicates that $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$, i.e. , $\boldsymbol{Q}^*$ cannot be reached by an admissible path. Alternatively, $\boldsymbol{Q}^*$ may be a point in $\partial(\partial(\mathcal{M}'))$ or a point very close to this set.

As in Section 5.1, the PPR-functions are given by (5.1). The chosen parameter values of the example are listed in Table 5. The objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (2.5) is used. Thus, we use Lagrange multipliers to find $\boldsymbol{Q}^*$, which can be found using (5.2). As in Section 5.1 we are interested in comparing $\boldsymbol{Q}^*_L$ with the boundary point $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ obtained using Algorithm 4.1, so we calculate $\boldsymbol{b}^* \in \mathcal{B}^w_1$. We consider one example, and Table 1 lists the coordinates of $\boldsymbol{Q}^*_L$ and $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$. Table 2 correspondingly lists the objective function values for $\boldsymbol{Q}^*_L$ and $\boldsymbol{Q}^*_{\boldsymbol{w}_1^*}$ and the Euclidian distance between these two points. In this case we see that the distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ is greater than in the previous example. Table 3 lists the weights of the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$.

As we can see from Table 3, the weight of reservoir 5, w_5^*, is significantly larger than the other weights. reservoirs. When we use Algorithm 4.1 to calculate the second-order fixed-weight strategy where reservoir 5 is given strict priority, denoted $\boldsymbol{w}_2^*$, we find that $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_2^*)) = \phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$, as can be read from Table 3. For all other higher-order fixed-weight strategies we obtain that $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_k^*)) < \phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$. Thus, the optimum is a boundary point of $\partial(\mathcal{M}')$ and Algorithm 4.1 managed to find it among the first-order fixed-weight strategies. This result is also consistent with the failure of the backtracking algorithm to find an admissible production strategy to reach $\boldsymbol{Q}_L^*$. Thus $\boldsymbol{Q}_L^*$ represents an inadmissible boundary point, i.e., $\boldsymbol{Q}_L^* \in \partial(\mathcal{M} \setminus \mathcal{M}')$.

5.3 A case where the optimal state is hard to find

In the final example neither steps of the two step optimization algorithm developed in Huseby & Haavardsson (2008) can be executed. The optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs is hard to find. Thus, the execution of the second step becomes difficult, since it assumes that Step 1 is done.

The theory in Huseby & Haavardsson (2008) puts restrictions on the PPR-functions of a specific field. In particular all the PPR-functions are assumed to be either convex or concave. In many examples the PPR-functions of some reservoirs are concave, and others are convex. Furthermore, the concave PPR-functions may be described by different classes of functions. Finally, some PPR-functions may be convex for some values of Q and concave for other values of Q. All these eventualities may be handled by the proposed parametric class.

Consider an examples where a field consists of 8 reservoirs. Four of the reservoirs may be described by linear PPR-functions, while the remaining four can be described by concave PPR-functions. For the first four reservoirs the PPR-functions are given by

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)),\ i = 1, \ldots, 4, \tag{5.3}$$

where $V_1, \ldots, V_4$ denote the recoverable volumes from the 4 reservoirs and D_i is the *scale parameter* of reservoir i. For the four remaining reservoirs the PPR-functions are given by (5.1). The chosen parameter values of the example are listed in Table 5.

The theory in Huseby & Haavardsson (2008) states that if all the PPR-functions are linear a specific $n-$th priority strategy is optimal with respect to a wide class of objective functions. In this case we may not apply this theory directly, since some of the PPR-functions are linear and other concave. The method of Lagrange multipliers may be used numerically or analytically if the optimum is an interior point of $\partial(\mathcal{M}')$. If the optimum is a point of the boundary of $\partial(\mathcal{M}')$, Lagrange multipliers may not be used.

As described in Section 4.2 we start out using Algorithm 4.1 to find the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$, which is displayed in Table 3. An inspection of $\boldsymbol{w}_1^*$ indicates that reservoir $5, 6, 7$ and 8 receive far higher priorities than the other reservoirs. This might indicate that the optimal production strategy reaches the boundary of $\partial(\mathcal{M}')$. Consequently we are interested in examining higher-order strategies, as described in Section 4.2. Examination among the $n-1 = 8-1 = 7$ relevant partitions of second order strategies assigning the highest priority to reservoir 5 we find that the objective function ϕ_K indeed increases when we search among second order strategies. Carrying on we obtain improvements until we reach fifth order strategies denoting the resulting optimum candidate $\boldsymbol{w}_5^*$. Table 3 lists the weights of $\boldsymbol{w}_5^*$ and the corresponding ordered partition $\{A_j\}_j^5$. From Table 3 we see that ϕ_K is increased by 1.2% compared with $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$. In this example we did not obtain any further improvement in ϕ_K by searching among even higher order strategies, i.e., among sixth, seventh and eighth order strategies. Thus we conclude that fifth order strategies represent the correct order and that $\boldsymbol{w}_5^*$ is optimal. To illustrate that further improvement could not be obtained by searching among higher order strategies we compare $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_5^*))$ with the best eighth order priority strategy, denoted $\boldsymbol{\pi}^*$. This strategy is a strict priority strategy and we find that $\{A_j\}_j^8 = \{5\}, \{6\}, \{7\}, \{8\}, \{2\}, \{0\}, \{1\}, \{3\}$. Furthermore, we find that $\phi_K(\boldsymbol{b}^*(\boldsymbol{\pi}^*)) = 30,926$ kSm^3, so $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_5^*))$ is 4.2 % larger than $\phi_K(\boldsymbol{b}^*(\boldsymbol{\pi}^*))$. Note also that the performance of $\boldsymbol{\pi}^*$ is significantly inferior to the performance of $\boldsymbol{w}_1^*$.

When the number of reservoirs is fairly small, say $n \leq 6$, we have seen that Algorithm 4.1 manages to find good solutions among the first-order fixed weight production strategies even when the optimal $\boldsymbol{Q}^*$ belongs to the boundary of $\partial(\mathcal{M}')$. This occurred in the example of Section 5.2. However, this breaks down in higher dimensional examples, as demonstrated in the present example.

6 The modelling of uncertainty

6.1 Including uncertainty in the model

We will now describe how robustness and sensitivity analyzes of an optimal production strategy $\boldsymbol{b}^*$ can be carried out, where $\boldsymbol{b}^*$ is found using the approach explained in Section 4.2. The robustness and sensitivity analysis is typically run *before* any production starts. The purpose is to discover how vulnerable the optimal strategy is when exposed to uncertainty. If $\boldsymbol{b}^*$ is very vulnerable to uncertainty, perhaps a more robust production strategy should be selected.

In this paper we will add uncertainty to the framework using the approach developed in Haavardsson & Huseby (2007), where a Monte Carlo simulation approach is used in the stochastic simulation. Uncertainty is added to the production model by modelling some of the key parameters as stochastic variables. A large sample, N, of the key parameters is generated, and every simulated vector of key parameters produces one simulated production profile. Using this approach, we obtain a sample of N simulated production profiles. A Monte Carlo simulation of the production can be done using Algorithm A.1 stated in Appendix A.

If we were to add uncertainty to the framework utilizing the framework developed in Huseby & Haavardsson (2008), we would use the constructed two-step optimization algorithm, where the second step is solved using a backtracking algorithm. Then we would use the Monte Carlo approach described in Algorithm A.1. A natural approach would be to apply the backtracking algorithm on a base case, i.e., a case that expressed the expected values of the stochastic variables. Denote the optimum for the base case $\boldsymbol{Q}^e \in \partial(\mathcal{M}^{'})$. Then we would obtain an admissible production strategy $\boldsymbol{b}^e \in \mathcal{B}^{'}$, i.e., an admissible path from $\boldsymbol{Q}^e$ back to $\boldsymbol{0}$, assuming that the second step involving backtracking may be successfully solved. A natural next step would be to use the Monte Carlo sampling technique described in Algorithm A.1 to create N simulated objective functions.

The backtracking works in a deterministic model where all parameters are known. For every point in time we then know how to produce every reservoir, because the proportions between the different reservoirs are known and the backtracking algorithm has found an admissible path, based on these proportions. When uncertainty is added these proportions will be distorted, and we cannot be guaranteed that $\boldsymbol{b}^e \in \mathcal{B}'$, i.e., the production strategy that yielded an admissible path from $\boldsymbol{Q}^e$ back to $\mathbf{0}$ in the base case, produces an admissible path when uncertainty is added. In fact, it is not obvious at all how the production strategy found with a deterministic model should be interpreted when uncertainty is added.

Using the proposed parametric class of the present paper we obtain an admissible production strategy for every sample of stochastic recoverable volumes and start rates as specified in Algorithm A.1, which is clearly very advantageous.

6.2 Robustness and sensitivity analysis

The algorithm below describes how the robustness and sensibility analysis is executed.

Algorithm 6.1

STEP 1. *Use ordinary differential equations and multi-segmented models as explained in Appendix A and Haavardsson & Huseby (2007) to create a vector of PPR-functions* $\boldsymbol{f}(\boldsymbol{Q}(t))$.

STEP 2. *Use Algorithm 4.1 to find the production strategy* $\boldsymbol{b}^* \in \mathcal{B}_1^w$ *that maximizes* $\phi_{K,0} = \phi_K$ *numerically, where* $\phi_{K,0} = \phi_K$ *is defined in* (2.5).

STEP 3. *Use the Monte Carlo sampling technique described in Algorithm A.1 to create N simulated objective functions* $\phi_K^j(\boldsymbol{b}^*) = \sum_{i=1}^n Q_i^j(T_K(\boldsymbol{b}^*)) = KT_K^j(\boldsymbol{b}^*), \quad j = 1, \ldots, N$.

Note that the vector $\boldsymbol{f}(\boldsymbol{Q}(t))$ in Step 1 is a vector of simplified production profile models, i.e., a curve fit of the vector of deterministic production models generated in the reservoir simulator.

We will assume that the recoverable volumes and the start rates of the reservoirs are stochastic. Since the start rates can be predicted with a high degree of certainty from e.g. well tests, we will assume that the uncertainty associated with the recoverable volumes is far greater than the uncertainty associated with the start rates. The sensitivity analysis will give us a variation in the plateau length T_K as a function of the decline rates, since $\phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. If the variability in T_K is great compared to the gain in plateau volume we achieve by using the optimizing techniques, this is not so good. The expected gain obtained using the optimization should be considerable. If the variability of T_K using $\boldsymbol{b}^*$ is great compared to the variability using other selected production strategies, it is relevant to ask whether the more robust production strategies should be selected.

Inspired by the Sharpe ratio used in portfolio analysis, see Sharpe (1994), we will propose a measure to compare production strategies. The Sharpe ratio is a measure of the mean excess return per unit of risk in an investment asset or a trading strategy and is defined as:

$$S = \frac{E[R - R_f]}{\sigma} = \frac{E[R - R_f]}{\sqrt{\{\mathrm{Var}[R - R_f]\}}},$$

where R is the asset return, R_f is the return on a benchmark asset, such as the risk free rate of return, $E[R - R_f]$ is the expected value of the excess of the asset return over the benchmark return, and σ is the standard deviation of the excess return.

In our situation selected production strategies play the roles of the assets. We will compare the performance of the n-th order priority strategies, defined in Section 3.1, and the production strategy obtained using Algorithm 4.1. The symmetry production strategy, defined in Section 3, will be used as a benchmark production strategy. The production strategies will then be compared to the symmetry production strategy using our version of the Sharpe ratio, referred to as the *performance ratio*:

$$P_i = \frac{E[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]}{\sqrt{\{\mathrm{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]\}}}, \tag{6.1}$$

where $\phi_K(\boldsymbol{b}^i)$ is the value of the objective function using production strategy i for a selection of production strategies $i = 1, \ldots, M$, where $M = n! + 1$ in this paper. The moments $E\{\phi_K(\boldsymbol{b}^i)\}$ and $\sqrt{\{\mathrm{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]\}} = \sqrt{\{\mathrm{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^M)]\}}$ are estimated empirically using the simulations. The optimal production strategy should ideally come out best most frequently in the simulations. Thus we compare the frequency at which each production strategy is best-performing during all the simulations.

6.2.1 The uncertainties used in the example

The framework described above will now be demonstrated in an example. Table 4 displays P10 and P90, i.e. the 10 percentile and the 90 percentile in the distributions of the stochastic producible volumes of the examples.

	Reservoir 1		Reservoir 2		Reservoir 3		Reservoir 4	
Example	P10	P90	P10	P90	P10	P90	P10	P90
Section 6.2.2	84 %	117 %	81 %	121 %	74 %	128 %	80 %	121 %

Table 4: *The $P10$ and $P90$ of the stochastic distributions of the producible volumes in the example. 100 % refers to the expected value, which is selected to be the deterministic producible volume.*

6.2.2 An example with concave PPR-functions

We consider an example with four reservoirs, where the *multi-segmented* PPR-functions $\{f_1, f_2, f_3, f_4\}$ are given by

$$f_{i,j}(Q_i(t)) = D_{i,j}(V_{i,j} - (Q_i(t) - \sum_{k<j} V_{i,k})), \quad i = 1, 2, 3, 4 \quad j = 1, 2, 3 \tag{6.2}$$

where $D_{i,j}$ denotes the *scale parameter* of the j-th segment of the i-th reservoir. We assume that $D_{i,1} \leq D_{i,2} \leq D_{i,3}$ for $i = 1, 2, 3, 4$. For an introduction to multi-segmented production profiles, see Appendix A or Haavardsson & Huseby (2007). The parameter values of the PPR-functions are given in Table 6. We let $K = 5.0$ kSm^3 per day.

We then proceed to Step 2 of Algorithm 6.1, where we use Algorithm 4.1 to find the candidate $\boldsymbol{b}^* \in \mathcal{B}_1^w$ that the numerical algorithm proposes as the best production strategy. Thus we obtain that $\boldsymbol{w}_1^* = (1.706, 0.575, 0.399, 1.0)$. Finally, we perform Step 3 of Algorithm 6.1 by simulating $N = 5,000$ objective functions for M selected production strategies. We consider the n-th order priority strategies and the strategy $\boldsymbol{b}^* = \boldsymbol{b}^*(\boldsymbol{w}_1^*)$, so we let

$M = n! + 1 = 4! + 1 = 24 + 1 = 25$. The performance ratio P_i defined by (6.1) may now be estimated for the 25 strategies.

Figure 3 shows results from the simulations. In the left panel we see the frequency at which every production strategy is best-performing, indicated with the columns in the graph. The frequencies can be read from the left axis in the graph. The blue curve in the left panel shows the performance ratio P_i of each production strategy, which can be read from the right axis of the graph in the same panel. The red curve, also relating to the right axis of the graph in the left panel, shows the performance of each production strategy in the deterministic case relative to $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$ in the deterministic case. We see that $\boldsymbol{b}^*$ is optimal in the deterministic case and remains the best-performing also when uncertainty is introduced in this example. The best-performing frequency of each production strategy is reconcilable with the magnitude of each performance ratio P_i.

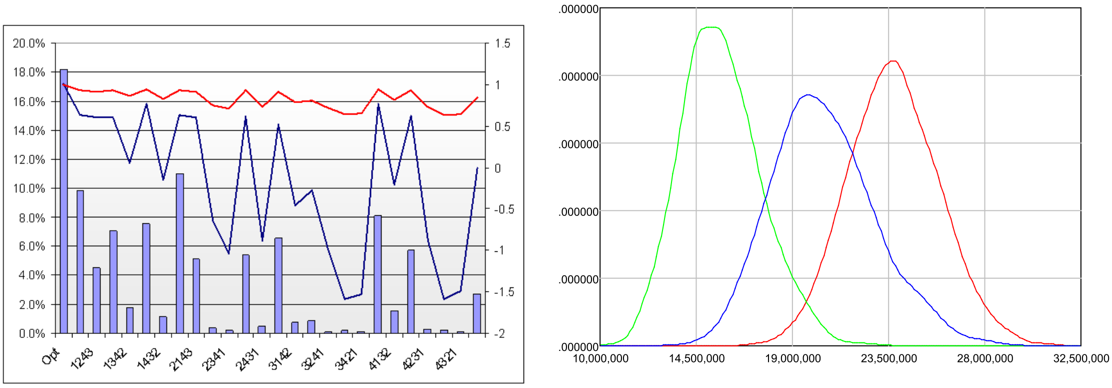

Figure 3: *The left panel shows every production strategy's best-performance frequency, performance ratio and ϕ_K in the deterministic case relative to $\phi_K(\boldsymbol{b}^*)$. The right panel shows the estimated densities of the plateau production ϕ_K of the strategy that is optimal in the deterministic situation (red graph), the symmetry strategy (blue graph), and the n-th order priority strategy $\boldsymbol{\pi} = \{3, 4, 1, 2\}$ (green graph).*

7 Conclusions

In the present paper we have introduced a parametric class of admissible production strategies referred to as *fixed-weight strategies*. Such strategies are stable, robust solutions that are easy to interpret. Thus, the production rates can be used in long-term planning of the actual production of a field consisting of many reservoirs. Compared to the strategies obtained using the two-step algorithm proposed in Huseby & Haavardsson (2008), fixed-weight strategies are also better suited for feedback to the reservoir simulation team on possible modifications of the simulations.

In cases where the first step of the algorithm proposed in Huseby & Haavardsson (2008) can be handled analytically, this method is extremely fast having a simulation time which grows linearly in the number of reservoirs. Compared to this, finding the optimal fixed-weight strategy is not as numerically efficient. Since, however, the number of parameters needed to define a fixed-weight strategy, is bounded by the number of reservoirs, complex, high-dimensional examples can easily be handled. Hence, the efficiency of this method is sufficient for most applications.

We have also demonstrated how uncertainty can be added into the proposed framework. This enables robustness and sensitivity studies of different production strategies. The performance criterion gives an indication of how robust every strategy is when exposed to uncertainty.

Under mild restrictions on the objective function it can be shown that an optimal production strategy can be found within the class of fixed- weight strategies. We will return to this important issue in a forthcoming paper.

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A A brief introduction to multi-segmented production profiles using ordinary differential equations

Single Arps curves, introduced by Arps (1945) model the production rate function and the cumulative production function mathematically through a one-way, causal relation. In Haavardsson & Huseby (2007) this approach is extended to multiple segments so that a combination of Arps curves may be used to get a satisfactory fit to a specific set of production data.

To also take into account various production delays, the dynamic two-way relation between the production rate function and the cumulative production is modelled in terms of a differential equation. The relation between the production rate function, q, and the cumulative production function, Q, should be of the following form:

$$q(t) = f(Q(t)), \qquad \text{for all } t \geq 0, \tag{A.1}$$

with $Q(t_0) = 0$ as a boundary condition.

The differential equation approach can also be extended to the more general situation where the production rate function consists of s segments. For each segment we assume that we have fitted a model in terms of a differential equation on the form given in (A.1). In order to connect these segment models, we need to specify a *switching rule* describing when to switch from one segment model to the next one. We define a switching rule based on the produced volume. By using this switching rule, we obtain a model for the combined differential equation.

A Monte Carlo simulation of the production can be done using the following procedure:

Algorithm A.1

[2]Mr. Wickmann graduated from University of Oslo in 2007.

STEP 1. *Assume that a production profile is divided in s segments. Generate* $V_1, \ldots, V_s$ *using the specified joint distribution* $p(V_1, \ldots, V_s)$ *for* $V_1, \ldots, V_s$*, where* V_i *denotes the producible volume of segment* i, $\quad i = 1, \ldots, s$.

STEP 2. *Generate* $r_0, r_1, \ldots, r_s$ *using the specified conditional joint distribution.* $p(r_0, r_1, \ldots, r_s | V_1, \ldots, V_s)$ *for the rates at the segmentation points* $r_0, r_1, \ldots, r_s$*, given the segment volumes.*

STEP 3. *Calculate* $D_1, \ldots, D_s$.

STEP 4. *Generate* t_0*, which may be subject to uncertainty related to the progress of the development project, drilling activities etc. Thus, one will typically assess a separate uncertainty distribution for this quantity. Then calculate* $t_1, \ldots, t_s$.

STEP 5. *Calculate* $q(t)$ *and* $Q(t)$.

B Descriptions of reservoirs used in examples

Example	Reservoir	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate $\sqrt{D_i V_i}$ (kSm^3/d)
Section 5.1	1	4.0	0.28	1.5
	2	5.0	0.40	2.0
	3	7.0	1.14	4.0
Section 5.2	1	4.0	0.28	1.5
	2	5.0	0.62	2.5
	3	7.0	2.57	6.0
	4	6.0	1.69	4.5
	5	8.0	0.39	2.5
	6	9.0	1.99	6.0
Section 5.3	1	4.0	0.0781	4.6
	2	4.0	0.0782	4.8
	3	4.4	0.0776	6.0
	4	4.4	0.0892	7.0
	5	4.0	0.0311	0.8
	6	5.0	0.0512	2.5
	7	3.0	0.0731	3.0
	8	7.0	0.0913	12.5

Table 5: *Parameter values for the examples in Section 5.*

		Segment 1		Segment 2		Segment 3		
Reservoir	Total reserves (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Stop rate (kSm^3)
1	10,000	7,000	3.0	1,800	1.9	1,200	1.3	0.01
2	6,000	3,600	2.6	1,320	2.1	1,080	1.1	0.01
3	7,000	5,460	5.0	1,001	3.1	539	2.2	0.02
4	4,000	3,080	3.0	570	1.9	350	1.3	0.01

Table 6: *Parameter values for the four reservoirs used in Section 6.2.2.*

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